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Distortion of a quantum field by point and line boundaries

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Abstract. The local distortion of a quantum scalar field $\hat{\phi}$ by isolated point and (straight-) line boundaries is studied using simple image-charge constructions. Non-zero temperature is considered. The distortion (away from uniformity) of the $\hat{\phi}$ virtual-particle sea and thermal gas by these boundaries is calculated exactly. Periodic arrays of point and line boundaries are as easily dealt with.

1. Introduction

In quantum field theory a spatial boundary surface ∂m on which the quantum field $\hat{\phi}$ must satisfy some specific boundary condition (say $\hat{\phi} = 0$ or $\partial_\perp \hat{\phi} = 0$) is qualitatively distinguished from all locations in space where $\hat{\phi}$ satisfies no such condition. The presence of ∂m causes the $\hat{\phi}$ vacuum or virtual-particle sea (which in free spacetime would of course be uniform) to become non-uniform—literally a function of the distance to ∂m [1–16]. Far from ∂m the quantum field and its vacuum are practically what they would be in free space. However, the distortion of $\hat{\phi}$ and its vacuum increases without limit as the boundary is approached. This is clearly revealed by the $\hat{\phi}$ local quantum functions; e.g. the $\hat{\phi}$ vacuum energy density. A local vacuum function having dimension $(\text{mass})^{\mathrm{d}}$ will, barring cancellations, diverge like $\delta^{-\mathrm{d}}$ as the distance $\equiv \delta$ to ∂m goes to zero [1–16]. In the past not all workers have viewed these boundary divergences as physical. However, they are physically meaningful. Their role is to express in mathematical terms the vacuum distortion caused by the boundary, which as already mentioned increases without limit as $\delta \to 0$.

In this paper we investigate the distortion of a scalar field $\hat{\phi}$ by isolated point and line boundaries on which $\hat{\phi}$ is constrained to satisfy specific (Dirichlet or Neumann) boundary conditions. To the author's knowledge these problems have not previously been investigated. They are easily solved by simple image-charge constructions of the appropriate heat kernels. Because not everyone is familiar with boundary divergences, we briefly review some comparable results for planar boundaries.

For a planar boundary ∂m constraining a massless quantum field $\hat{\phi}$, simple dimensional arguments predict the form boundary divergences must have. In four spacetime dimensions [3]

$$\langle T_{00}(x)\rangle = \frac{C_1}{\delta^4} \qquad \langle|\hat{\phi}(x)|^2\rangle = \frac{C_2}{\delta^2}\cdots \tag{1.1}$$

where δ is the perpendicular distance from the field point to ∂m. Of course the dimensionless constants $C_{1,2}$ have to be computed. One finds $C_1 = -C_2 = \pm(4\pi)^{-2}$ for Dirichlet/Neumann conditions. In free spacetime these constants must vanish because there is no dimensional parameter available in terms of which functions like $\langle T_{00}(x)\rangle$ and

$\langle|\hat{\phi}(x)|^2\rangle$ can be expressed. However, the introduction of ∂m makes available the length δ in terms of which these vacuum functions can be expressed. Almost inevitably, boundary divergences appear. Non-zero mass modifies but does not eliminate the behaviour (1.1). Cancellation seems to be the only way for boundary divergence *not* to be present in quantum field theory. For an electromagnetic field excluded from half of space by a planar metallic boundary, *the boundary divergences of the electric and magnetic sectors exactly cancel* [1, 2]. However, as soon as the metal plane is given curvature, this cancellation stops being perfect [3].

The argument leading to equations (1.1) assumed the planar boundary ∂m is infinitely far from any other spatial structure. Then there is no global Casimir effect. A second parallel boundary a finite distance L away would complicate the argument by introducing the constant length parameter L. A second boundary of any shape introduces at least one such parameter. In situations like these, boundary divergences are still present. Moreover, they contribute to the global Casimir effect [14, 15]. One expects a global vacuum energy shift whenever a global length parameter is available. (There are exceptions to this, however, as discussed in section 4.)

In section 2 we discuss the simplest boundaries having cylindrical and spherical symmetry. These are straight-line boundaries and point boundaries, which one can regard as $R \to 0$ limits of circular cylinders and spheres of radius R. To do the $R > 0$ problems completely one would need a complete set of quantum modes appropriate for each. These modes contain Bessel function factors, making local mode-sum calculations more elaborate than the relatively simple ones for rectangular boundary geometry. Fortunately these local calculations are manageable. (See [11, 12] for explicit local analysis of spherical cavities, and also the global calculations [17–19].) Local analysis for circular cylinder geometry seems to be lacking. (See, however, [20] for a detailed global calculation.)

Consider the Casimir problem in the space exterior to vanishingly small isolated circular cylinders and spheres, on whose surfaces the modes of a massless scalar quantum field $\hat{\phi}$ must satisfy either Dirichlet or Neumann conditions. Because R is not available we have again a situation like the one leading to equations (1.1); the only length parameter available is the distance r from the line or point boundary to field point x. Thus

$$\langle T_{00}(x)\rangle = \frac{B_1}{r^4} \qquad \langle|\hat{\phi}(x)|^2\rangle = \frac{B_2}{r^2}\cdots \tag{1.2}$$

where $B_{1,2}$ are the dimensionless constants which must be computed. While this can be done by means of explicit mode sums (as will be shown elsewhere) the answer can be obtained far more easily using simple image-charge arguments (which effectively perform mode sums) to construct the exact spatial heat kernels $\langle x|\exp(t\Delta)|y\rangle$ for the problems considered.

Calculations in this paper are done by the local ζ-function method—the version described in [16] for Euclidean spacetime. Once $\langle x|\exp(t\Delta)|x\rangle$ is known, it is straightforward to evaluate the local energy density of the vacuum and various other vacuum functions. First one calculates the local ζ-function for the zero-point fluctuations of the scalar field. For non-zero mass M and temperature T this ζ-function is

$$\begin{aligned} Z_\beta(s|x) &\equiv Z_{\rm sea}(s|x) + E^{\beta}_{\rm gas}(s|x) \\ &\equiv \frac{\mu^{2s}}{\Gamma(s)}\int_0^\infty {\rm d}t\, t^{s-1}{\rm e}^{-tM^2}\langle x|{\rm e}^{t\Delta}|x\rangle(4\pi t)^{-1/2}\left[1+2\sum_{m=1}^{\infty}{\rm e}^{-(m\beta)^2/4t}\right] \end{aligned} \tag{1.3}$$

where $\beta = 1/T$ and μ is an arbitrary mass parameter. Henceforth we set $\mu = 1$ because it plays no role in this paper. The square bracket in equation (1.3) is the factor in the heat kernel representing x_0, the imaginary time direction. The sum $\sum_m$ expresses the discreteness of the x_0s conjugate momentum k_0 for $T > 0$. In the limit $T \to 0$ the square bracket approaches unity, which corresponds to k_0 becoming continuous. The ζ-function $Z_{\rm sea}(s|\boldsymbol{x})$ is the T-independent part of equation (1.3) and it represents the virtual particle sea of the quantum field. The effective potential density of this sea is

$$\mathcal{L}_{\rm sea}(\boldsymbol{x}) \equiv -\frac{\rm d}{{\rm d}s} Z_{\rm sea}(s|\boldsymbol{x})|_{s=0}. \tag{1.4}$$

The T-independent part of the Casimir effect resulting from the distortion of the virtual sea is described entirely in terms of $\mathcal{L}_{\rm sea}(\boldsymbol{x})$. The function $E^{\beta}_{\rm gas}(s|\boldsymbol{x})$ (which is not a ζ-function) is entirely T-dependent. It represents the thermal gas associated with the $T > 0$ scalar field. In the limit $T \to 0$ this function vanishes. From it one obtains the thermodynamic potential *density* of the thermal gas

$$\mathcal{L}^{\beta}_{\rm gas}(\boldsymbol{x}) \equiv -\frac{\rm d}{{\rm d}s} E^{\beta}_{\rm gas}(s|\boldsymbol{x})|_{s=0} \qquad \Omega = -PV = \int {\rm d}^3 x \mathcal{L}^{\beta}_{\rm gas}(\boldsymbol{x}). \tag{1.5}$$

Another vacuum function it is interesting to consider is (see also [6, 15])

$$\langle|\hat{\phi}(\boldsymbol{x})|^2\rangle = Z_{\beta}(1|\boldsymbol{x}). \tag{1.6}$$

2. Isolated line and point boundaries

The simple image-charge device employed here for line and point boundaries must have occurred to others previously. It is, however, not standard. Image charges are typically used one-dimensionally to implement boundary conditions on planes, or in two and three dimensions to do the same on cylindrical and spherical surfaces. The author has not seen them used to implement boundary conditions on isolated lines and points, as will be done here.

To begin, let us recall the form of the heat kernel of the Laplace operator $(-\Delta)$ on flat, boundaryless n-dimensional space E^n;

$$k_n(t|\boldsymbol{x}-\boldsymbol{y}) = k_n(t|\boldsymbol{y}-\boldsymbol{x}) = \langle \boldsymbol{x}|{\rm e}^{t\Delta}|\boldsymbol{y}\rangle = (4\pi t)^{-n/2}{\rm e}^{-(\boldsymbol{x}-\boldsymbol{y})^2/4t}. \tag{2.1}$$

Because

$$(-\Delta_x)k_n(t|\boldsymbol{x}-\boldsymbol{y}) = -\partial/\partial t k_n(t|\boldsymbol{x}-\boldsymbol{y}) = \left[\frac{n}{2t} - \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{4t^2}\right] k_n(t|\boldsymbol{x}-\boldsymbol{y})$$

k_n satisfies the heat equation $(-\Delta_x + \partial/\partial t)k_n(t|\boldsymbol{x}-\boldsymbol{y}) = 0$. It therefore represents a process of diffusion during a (fictitious) 'proper time' interval t from source point $\boldsymbol{y}$ to field point $\boldsymbol{x}$ through flat, boundaryless space E^n. In more general language the heat kernel is

$$\langle x|{\rm e}^{-tA}|y\rangle \equiv \sum_m {\rm e}^{-t\lambda_m}\Psi_m(x)\bar{\Psi}_m(y) \tag{2.2}$$

where $A\Psi_m(x) = \lambda_m\Psi_m(x)$ is the eigenvalue problem for some operator A defined on a manifold $\mathcal{M}$. The heat kernel's main virtues result largely from the absolute and uniform convergence of the mode sum (2.2) for $t > 0$. This enables $\langle x|\exp(-tA)|y\rangle$ to satisfy exactly the same boundary conditions as do the individual modes $\Psi_m(x)$. Being a solution of the heat equation, the function (2.2) represents some kind of diffusion process through $\mathcal{M}$ from y to x in proper time t. If one is able to construct a heat kernel satisfying the correct heat equation and boundary conditions, then one has the correct heat kernel.

Concerning equation (2.1), note that $k_n(t|\boldsymbol{x} \pm \boldsymbol{y} + \boldsymbol{C})$ satisfies for any constant vector $\boldsymbol{C}$ the heat equation $(-\Delta_x + \partial/\partial t)k_n = 0$. This will be very important for our later use of image charges.

Now let us imagine spacetime being Euclidean and N-dimensional, $E^N = E^{n_1} \times E^{n_2}$ with $N = n_1 + n_2$. Integer n_1 will represent a number of free, unbounded dimensions while n_2 will be a number of dimensions in which we assume a point boundary at the centre. Thus, for line and point boundaries in $N = 4$ dimensional spacetime, $n_1 = 2$, $n_2 = 2$ and $n_1 = 1$, $n_2 = 3$ respectively. However, one can just as easily calculate with arbitrary $n_{1,2}$, and why not do so? Heat kernels on factorized spaces themselves factorize, and so for E^N with no boundaries

$$\langle x|e^{t\Delta}|y\rangle_{\text{free}} = k_{n_1}(t|\boldsymbol{x}_1 - \boldsymbol{y}_1)k_{n_2}(t|\boldsymbol{x}_2 - \boldsymbol{y}_2) \tag{2.3}$$

where $x = (\boldsymbol{x}_1, \boldsymbol{x}_2)$ and $y = (\boldsymbol{y}_1, \boldsymbol{y}_2)$.

2.1. Dirichlet point at $\boldsymbol{x}_2 = 0$

In E^{n_1} free conditions are assumed and the modes are just plane-wave factors with continuous momentum. In E^{n_2} the modes are assumed to vanish at $\boldsymbol{x}_2 = 0$. There are no other boundary conditions. The heat kernel must vanish at $\boldsymbol{x}_2 = 0$ or $\boldsymbol{y}_2 = 0$ with no other boundary conditions. It is

$$\langle x|e^{t\Delta}|y\rangle_{\text{D}} = k_{n_1}(t|\boldsymbol{x}_1 - \boldsymbol{y}_1)[k_{n_2}(t|\boldsymbol{x}_2 - \boldsymbol{y}_2) - k_{n_2}(t|\boldsymbol{x}_2 + \boldsymbol{y}_2)] \tag{2.4}$$

which clearly has the correct boundary behaviour. From our previous discussion the function (2.4) is also clearly a solution of the Laplace operator's heat equation. Equation (2.4) is much like the image charge construction of the heat kernel for a field confined to half of space by a Dirichlet plane. (This problem corresponds, in $N = 4$ dimensions, to $n_1 = 3$, $n_2 = 1$.) What occurs in E^{n_2} is diffusion from a positive source at $\boldsymbol{y}_2$ toward field point $\boldsymbol{x}_2$, together with diffusion from a negative source at $-\boldsymbol{y}_2$. The result is a heat kernel with vanishing behaviour at the midway *point* $\boldsymbol{x}_2 = 0$ rather than vanishing behaviour on the plane midway between the sources.

To obtain physical quantum densities we set $x = y$:

$$\langle x|e^{t\Delta}|x\rangle_{\text{D}} = (4\pi t)^{-N/2}[1 - e^{-r^2/t}] \qquad r^2 \equiv \boldsymbol{x}_2 \cdot \boldsymbol{x}_2. \tag{2.5}$$

The term $(4\pi t)^{-N/2}$ represents the uniform free spacetime contribution of equation (2.3). We are much more interested in the boundary part, whose term in equation (1.3) is

$$Z_{\text{B}}(s|x) \equiv \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1}(4\pi t)^{-N/2}[-e^{-r^2/t}] = -\frac{r^{2s}\Gamma(-s+N/2)}{(4\pi)^{N/2} r^N \Gamma(s)} \qquad M = 0 \tag{2.6a}$$

$$= -\frac{2}{(4\pi)^{N/2}\Gamma(s)} \left(\frac{r}{M}\right)^{s-N/2} K_{-s+N/2}(2Mr) \qquad M > 0. \tag{2.6b}$$

In equation (2.6*a*) we use

$$r^{2s}\Gamma(-s) = \int_0^\infty dt\, t^{s-1} e^{-r^2/t} \tag{2.7}$$

while equation (2.6*b*) is an integral formula for the modified Bessel function $K_\nu(z)$:

$$2(M/r)^s K_{-s}(2Mr) = \int_0^\infty dt\, t^{s-1} e^{-tM^2} e^{-r^2/t}. \tag{2.8}$$

Analytic continuation is manifest in equations (2.6)–(2.8) and will be taken for granted. Note that $Z_{\text{B}}(s|x)$ has none of the properties of a true ζ-function. It is just being treated like one notationally.

From $Z_B(s|x)$ we obtain the effective action density or vacuum energy density (1.4);

$$\mathcal{L}_{\text{sea}}(\boldsymbol{x}) - \mathcal{L}_{\text{free}} = \frac{\Gamma(N/2)}{(4\pi)^{N/2} r^N} \qquad M = 0 \tag{2.9a}$$

$$= \frac{2}{(4\pi)^{N/2}} \left(\frac{M}{r}\right)^{N/2} K_{N/2}(2Mr) \qquad M > 0 \tag{2.9b}$$

where, incidentally, in $N = 4$ dimensions the uniform free-spacetime term is $\mathcal{L}_{\text{free}} = (M^2/4\pi)^2[\ln(M/\mu) - \frac{3}{4}]$.

For the thermal gas from equation (1.3) one finds

$$E^{\beta}_{\text{gas}}(s|\boldsymbol{x}) = \frac{2\Gamma(-s+N/2)}{(4\pi)^{N/2}\Gamma(s)} \sum_{m=1}^{\infty} \left\{ \left(\frac{m\beta}{2}\right)^{2s-N} - \left[\left(\frac{m\beta}{2}\right)^2 + r^2\right]^{s-N/2} \right\} \qquad M = 0 \tag{2.10a}$$

$$= \frac{4}{(4\pi)^{N/2}\Gamma(s)} \sum_{m=1}^{\infty} \left\{ \left(\frac{2M}{m\beta}\right)^{s-N/2} K_{N/2-s}(m\beta M) - \left[\frac{M}{[r^2+(m\beta/2)^2]^{1/2}}\right]^{s-N/2} K_{N/2-s}(2M[r^2+(m\beta/2)^2]^{1/2}) \right\} \qquad M > 0. \tag{2.10b}$$

Then from equation (1.5) the thermodynamic potential density can be written down.

Equation (1.6) provides the vacuum expectation value

$$\langle|\hat{\phi}(\boldsymbol{x})|^2\rangle - \langle|\hat{\phi}|^2\rangle_{\text{free}} = Z_B(1|\boldsymbol{x}) + E^{\beta}_{\text{gas}}(1|\boldsymbol{x}) \tag{2.11}$$

which also need not be written out separately.

2.2. *Neumann point at $\boldsymbol{x}_2 = 0$*

In E^{n_1} free conditions are still assumed, while in E^{n_2} we now require the modes $\Psi_{2m}(\boldsymbol{x}_2)$ to satisfy Neumann conditions at $\boldsymbol{x}_2 = 0$: $\nabla_2\Psi_{2m}(\boldsymbol{x}_2)$ vanishes at $\boldsymbol{x}_2 = 0$. There are no other boundary conditions. The appropriate heat kernel is

$$\langle x|e^{t\Delta}|y\rangle_N = k_{n_1}(t|\boldsymbol{x}_1 - \boldsymbol{y}_1)[k_{n_2}(t|\boldsymbol{x}_2 - \boldsymbol{y}_2) + k_{n_2}(t|\boldsymbol{x}_2 + \boldsymbol{y}_2)]. \tag{2.12}$$

Checking the boundary condition one readily verifies

$$\nabla_x[k_n(t|\boldsymbol{x}-\boldsymbol{y}) + k_n(t|\boldsymbol{x}+\boldsymbol{y})] = -\frac{1}{2t}[(\boldsymbol{x}-\boldsymbol{y})k_n(t|\boldsymbol{x}-\boldsymbol{y}) + (\boldsymbol{x}+\boldsymbol{y})k_n(t|\boldsymbol{x}+\boldsymbol{y})]$$

which vanishes at $\boldsymbol{x} = 0$ as it should. The diagonal heat kernel is now

$$\langle x|e^{t\Delta}|y\rangle_N = (4\pi t)^{-N/2}[1 + e^{-r^2/t}] \tag{2.13}$$

and the only difference between this and the Dirichlet one (2.5) is the different sign of the r-dependent term. Consequently equations (2.6), (2.9)–(2.11) with this sign change give the local Casimir and thermal gas results for a Neumann point boundary.

2.3. *Three spatial dimensions*

Because of their particular interest we write out the $N = 4$ cases separately for Dirichlet conditions. The boundary line lies along the x_3-axis. The boundary point is positioned at

$x = 0$. For the line $r^2 = x_1^2 + x_2^2$ and for the point $r^2 = x_1^2 + x_2^2 + x_3^2$. The local Casimir and thermal gas results are

$$\mathcal{L}_{\rm sea}(x) - \mathcal{L}_{\rm free} = \begin{cases} \dfrac{1}{(4\pi)^2 r^4} & M = 0 \qquad (2.14a) \\[2ex] \dfrac{2}{(4\pi)^2}\left(\dfrac{M}{r}\right)^2 K_2(2Mr) & M > 0 \qquad (2.14b) \end{cases}$$

$$\mathcal{L}^{\beta}_{\rm gas}(x) = \begin{cases} -\dfrac{\pi^2 T^4}{45} + \dfrac{2}{(4\pi)^2}\displaystyle\sum_{m=1}^{\infty}[r^2 + (m\beta/2)^2]^{-2} & M = 0 \qquad (2.15a) \\[2ex] \dfrac{4}{(4\pi)^2}\displaystyle\sum_{m=1}^{\infty}\left\{ -\left(\dfrac{m\beta}{2M}\right)^2 K_2(m\beta M) \right. & \\ \quad \left. + \dfrac{1}{M^2}[r^2 + (m\beta/2)^2] K_2(2m[r^2 + (M\beta/2)^2]^{1/2}) \right\} & M > 0 \qquad (2.15b) \end{cases}$$

$$\langle|\hat{\phi}(x)|^2\rangle - \langle|\hat{\phi}|^2\rangle_{\rm free}$$
$$= \begin{cases} -\dfrac{1}{(4\pi)^2 r^2} + \dfrac{T^2}{12} - \dfrac{2}{(4\pi)^2}\displaystyle\sum_{m=1}^{\infty}\left[r^2 + \left(\dfrac{m\beta}{2}\right)^2\right]^{-1} & M = 0 \ (2.16a) \\[2ex] -\dfrac{2}{(4\pi)^2}\left(\dfrac{M}{r}\right) K_1(2Mr) + \dfrac{4}{(4\pi)^2}\displaystyle\sum_{m=1}^{\infty}\left\{ \left(\dfrac{m\beta}{2M}\right) K_1(m\beta M) \right. & \\ \quad \left. -\dfrac{1}{M}\left[r^2 + \left(\dfrac{m\beta}{2}\right)^2\right]^{1/2} K_1(2M[r^2 + (m\beta/2)^2]^{1/2}) \right\} & M > 0. \ (2.16b) \end{cases}$$

Note in equation (2.14) the divergence $+(4\pi r^2)^{-2}$ as $r \to 0$ for M zero or non-zero. Similarly, the vacuum value (2.16) behaves like $-(4\pi r)^{-2}$ as $r \to 0$. Note also that $\mathcal{L}_{\rm gas}(x) \to -\pi^2 T^4/45$ as $r \to \infty$ which is the Stefan–Boltzmann law for a massless scalar $T > 0$ gas. For $M > 0$ the 'massive Stefan–Boltzmann law' is what remains at large r in equation (2.15b). Similarly the thermal expectation value of $\langle|\hat{\phi}|^2\rangle$ is what remains on the right in equation (2.16) for large r.

3. More on line boundaries

Non-trivial variants of the preceding results for isolated line boundaries are easily obtained. One assumes, for a long line boundary, free conditions in the direction of the line. Instead one may assume some other boundary condition. This does not interfere with the image-charge construction enforcing boundary conditions on the line. A variety of systems can be constructed by choosing Dirichlet or Neumann conditions on the line, and perpendicular to this direction Dirichlet or Neumann conditions on planes, or periodic conditions, or something else. Explicit local Casimir and thermal gas results are easily obtained. For brevity we shall not write out most of these, but only indicate how derivations proceed.

To implement Dirichlet conditions in direction x at $x = 0, L$ the factor $(4\pi t)^{-1/2}\exp[-(x-y)^2/4t]$ in the heat kernel representing free conditions is replaced by

$$\sum(t, L|x, y)_{\rm D} = (4\pi t)^{-1/2} \sum_{n=-\infty}^{\infty} [e^{-(n2L+x-y)^2/4t} - e^{-(n2L+x+y)^2/4t}] \tag{3.1}$$

which vanishes at $x = 0, L$ or $y = 0, L$ as it should. For Neumann conditions one uses instead

$$\sum(t, L|x, y)_{\rm N} = \frac{1}{L} + (4\pi t)^{-1/2} \sum_{n=-\infty}^{\infty} [e^{-(n2L+x-y)^2/4t} + e^{-(n2L+x+y)^2/4t}]. \tag{3.2}$$

$\partial_x \sum_{\rm N}$ vanishes at $x = 0, L$ and similarly $\partial_y \sum_{\rm N} = 0$ at $y = 0, L$. Equations (3.1), (3.2) reduce, in the limit $L \to \infty$, to the factors appropriate for isolated Dirichlet and Neumann planar boundaries at $x = 0$.

Let us position an isolated Dirichlet boundary coincident with the (x_1x_2)-plane at $x_3 = 0$, and along the x_3-axis a Dirichlet line. The diagonal heat kernel (2.5) with $N = 4$ is replaced by

$$(\boldsymbol{x}|e^{t\Delta}|\boldsymbol{x}) = (4\pi t)^{-2}[1 - e^{-r^2/t}][1 - e^{-x_3^2/t}] \tag{3.3}$$

where $r^2 = x_1^2 + x_2^2$. Equation (2.14*a*) for $M = 0$ becomes

$$\mathcal{L}_{\rm sea}(\boldsymbol{x}) - \mathcal{L}_{\rm free} = \frac{1}{(4\pi)^2}[r^{-4} + x_3^{-4} - (r^2 + x_3^2)^{-2}]. \tag{3.4}$$

Equation (2.15*a*) becomes

$$\mathcal{L}^{\beta}_{\rm gas}(\boldsymbol{x}) + \pi^2 T^4/45 = \frac{2}{(4\pi)^2} \sum_{m=1}^{\infty} \{[r^2 + (m\beta/2)^2]^{-2} + [x_3^2 + (m\beta/2)^2]^{-2}$$
$$-[r^2 + x_3^2 + (m\beta/2)^2]^{-2}\}. \tag{3.5}$$

For a Neumann line (plane) all terms containing $r(x_3)$ change sign. If there are two planes perpendicular to x_3 one obtains exact results using the factors (3.1) or (3.2) in place of the x_3-dependent factor in equation (3.3). All variants including non-zero mass can be written down almost by inspection.

4. Periodic arrays of line and point boundaries

A simple extension of the construction for isolated line and point boundaries leads to exact local Casimir and thermal gas results for periodic arrays of such objects. These periodic arrays need not be rectangular; indeed they can be quite arbitrary arrangements characterized by arbitrarily many different (and individually arbitrary) vectors. The local distortion of the quantum field by such an array is clearly very complicated. Nonetheless, for symmetry reasons, there is no global Casimir energy shift for these systems.

4.1. Periodic arrays of parallel line boundaries

Generalizing the heat kernel (2.4) with $n_{1,2} = 2$ for an isolated Dirichlet line boundary (chosen to lie along the x_3-axis), let us introduce an arbitrary vector A lying in the (x_1x_2)-plane. The heat kernel

$$(4\pi t)^{-3/2} e^{-(x_3-y_3)^2/4t} \sum_{a=-\infty}^{\infty} [e^{-(aA+\boldsymbol{x}-\boldsymbol{y})^2/4t} - e^{-(aA+\boldsymbol{x}+\boldsymbol{y})^2/4t}]$$
$$\boldsymbol{x} = (x_1, x_2) \quad \boldsymbol{y} = (y_1, y_2) \tag{4.1}$$

vanishes when either of x or y equal mA, where $-\infty < m < \infty$ runs over all integers. Clearly this heat kernel represents an infinite periodic linear lattice of points in the (x_1x_2)-plane at $x = aA$, a = integer. These are points where an array of parallel equally-spaced Dirichlet lines puncture this plane. In three dimensions the scalar field vanishes on all of these lines.

The local Casimir and thermal gas description of this periodic array of parallel Dirichlet lines is obtained by inserting (4.1) in place of $\langle x|\exp(t\Delta)|x\rangle$ in equation (1.3). The results are just a sum over individual line boundaries, each contribution like equations (2.14)–(2.16) but centred at a different lattice point. For example, equation (2.14*a*) becomes

$$\mathcal{L}_{\rm sea}(x) = (4\pi)^{-2}\sum_{a\neq 0}\{[(\tfrac{1}{2}aA + x)^2]^{-2} - [(\tfrac{1}{2}aA)^2]^{-2}\} + (4\pi)^{-2}r^{-4} + \mathcal{L}_{\rm free}. \tag{4.2}$$

It is interesting that the exact local results (4.1), (4.2) can be extended from a single vector A to an arbitrary number of vectors $A_1, A_2, \ldots, A_N$ all lying in the (x_1x_2)-plane. Just replace aA in equations (4.1), (4.2) by $a_1A_1 + a_2A_2 + \ldots$ and sum individually $a_1, a_2, \ldots$ over all integers. The resulting mathematics represents for $N = 2$ a doubly-periodic arrangement of parallel line boundaries, and for $N > 2$ more complicated kinds of periodic arrays.

4.2. Periodic arrays of Dirichlet points

Generalizing the heat kernel (2.4) with $n_1 = 1$, $n_2 = 3$ for an isolated Dirichlet point, we introduce an arbitrary three-dimensional vector B. The heat kernel

$$(4\pi t)^{-3/2}\sum_{b=-\infty}^{\infty}[\mathrm{e}^{-(bB+x-y)^2/4t} - \mathrm{e}^{-(bB+x+y)^2/4t}] \tag{4.3}$$

vanishes at all points $x = mB$ where m is any integer. Clearly it represents a periodic infinite line array of Dirichlet points at which the scalar field vanishes. The formula corresponding to equation (4.2) is obvious. Indeed, all of the generalizations of equations (2.14)–(2.16) are easily obtained for the periodic array under discussion.

Equation (4.3) can be extended from an infinite periodic line array to an arbitrary periodic array characterized by N arbitrary three-dimensional vectors $B_1, B_2, \ldots$. Just replace bB by $b_1B_1 + b_2B_2 + \ldots$ and sum $b_1, b_2, \ldots$ individually over all integers.

One knows that a global Casimir force acts between two parallel planar boundaries. Presumably there exist such forces between parallel line boundaries, and between two point boundaries. The preceding results for infinite periodic arrays unfortunately do not tell us much about the Casimir force between two objects within the array. The reason is, the global Casimir effect vanishes for these arrays. There are at least two ways to understand this.

Consider the parallel line array. Any line will experience equal and opposite Casimir forces from both sides, and therefore be in equilibrium. If the array were pulled apart uniformly—i.e. if $A \to \lambda A$ with $\lambda > 1$ and increasing—then every line remains in equilibrium. No work is done. Conversely, beginning with $\lambda = \infty$, all lines are at infinity except the one at $(0, 0)$. Reducing λ from its initial large value down to $\lambda = 1$ will reconstruct the lattice, at no cost in energy.

A less elementary way to understand why the infinite array has no global Casimir effect is to observe that the result (4.2) for $\mathcal{L}_{\rm sea}(x)$ is simply the sum of individual line boundary functions over the entire array—nothing more. In the language of [14, 15], $\mathcal{L}_{\rm sea} = \mathcal{L}_{\rm B}$ is entirely 'boundary', with $\mathcal{L}_{\rm F} = 0$. When this is the case there is no global Casimir effect.

The comments just made about line arrays obviously extend to point arrays. This need not be discussed separately.

An infinite periodic array of parallel planar boundaries also has no global Casimir effect. Any two planes experience a mutual force. However, due to left–right symmetry, the net force on any plane is zero.

Whether one is considering a periodic array of points, lines or planes, it is clear that if just one object of the array were displaced, global Casimir forces on it and every other (not too distant) element of the array would come to life. Would these forces try to restore periodicity? This is one of many unexplored problems in Casimir theory.

Besides the Casimir effect there is also the thermal gas to consider. In equation (1.3), $E^{\beta}_{\mathrm{gas}}(s|x)$ satisfies for any s the same boundary conditions as does $\langle x|\exp(t\Delta)|x\rangle$. Thus $\mathcal{L}^{\beta}_{\mathrm{gas}}(x)$ and other local quantum functions characterizing the gas will vanish as any Dirichlet boundary is approached [16]. A Dirichlet boundary makes a hole in the gas, much as it makes a hole in the virtual sea. The vanishing of the right side of equation (2.15) at $r = 0$ illustrates this for an isolated point or line. Arrays of point and line boundaries compel the gas to vanish at every point or line in the array. The gas is highly non-uniform.

5. Conclusion

Imagine a quantum field with some number of boundaries $\partial m_1, \partial m_2, \ldots$ immersed in this field, constraining its modes. Each boundary is an intrinsically classical object. However, each boundary's distortion of the quantum field is quantum in nature. One might view (∂m_1+ its vacuum distortion), (∂m_2+ its vacuum distortion),... each as quantum 'objects'. Casimir theory is really the theory of such objects and how they interact with one another via the quantum field in which they exist. An array of metallic point, line or surface boundaries immersed in the quantum electromagnetic field would be such a system. These objects all interact through the electromagnetic field—not pairwise, but rather as a complicated many-body interaction. Casimir theory in fact originated [21] as an attempt to understand how atoms and structures built up from atoms (modelled as metal or dielectric objects) interact with and through the quantum electromagnetic field pervading all of space. This is surely a deep and interesting problem.

In this article the quantum electromagnetic field has been replaced by a quantum scalar field $\hat{\phi}$, and metal has been replaced by Dirichlet or Neumann boundaries. Our emphasis was on point and line boundaries and the distortion these cause in the $\hat{\phi}$ virtual particle sea and (for $T > 0$) thermal gas. New results for isolated point and line boundaries were obtained by a simple method. These calculations extend easily to infinite periodic arrays which, by symmetry, have no global Casimir forces. Corresponding results for other quantum fields should not be more difficult to obtain, but were not discussed here.

One would like to know more, of course. For example, how do two point boundaries interact? How do two line boundaries interact, or a point boundary with a line boundary? None of these two-object problems has yet been solved. Beyond these, there are more difficult many-body Casimir problems to investigate. Efforts to obtain at least approximate results for some of these systems are in progress.

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